MOVABLE SINGULARITIES OF SOLUTIONS OF DIFFERENCE EQUATIONS IN RELATION TO SOLVABILITY, AND STUDY OF A SUPERSTABLE FIXED POINT

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ABSTRACT. We overview applications exponential asymptotics and analyzable function theory to difference equations, in defining an analog of the Painlevé property for them and we sketch the conclusions with respect to the solvability properties of first order autonomous ones. It turns out that if the Painlevé property is present the equations are explicitly solvable and in the contrary case, under further assumptions, the integrals of motion develop singularity barriers. We apply the method to the logistic map $x_{n+1} = ax_n(1 - x_n)$ where it turns out that the only cases with the Painlevé property are $a = -2, 0, 2$ and 4 for which explicit solutions indeed exist; in the opposite case an associated conjugation map develops singularity barriers.

1. INTRODUCTION

We present an outline of new methods for determining the position and the type of singularities of a certain kind of solutions of difference equations [17] and use this information to perform Painlevé analysis on them. The approach relies on advances in exponential asymptotics and the theory of analyzable functions [10, 11, 23, 13, 14, 15, 16, 18, 20, 21, 25, 26, 34, 45]. The main concepts are discussed first.

Analyzable functions. Introduced by Jean Écalle, these are mostly analytic functions which at singular points are completely described by *transseries*, much in the same way as analytic functions are represented at regular points by convergent series. In contrast with analytic functions (which are not closed under division) and with meromorphic functions (which fail to be stable under integration and composition) analyzable ones are conjectured to be closed under all operations whence the grand picture of this theory, that all functions of natural origin are analyzable. In particular, solutions of many classes of differential and difference equations have been shown to be analyzable.

Transseries. Also introduced by J. Ecalle, transseries represent the "ultimate" generalization of Taylor series. Transseries are formal asymptotic combinations of power series, exponentials and logarithms and contain a wealth of information not only about local but also about global behavior of functions. One of the simplest nontrivial examples of a transseries is

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[1] In the sense stemming from Stokes original papers and the one favored in exponential asymptotics literature, *Stokes* lines are those where a small exponential is purely real; on an *antistokes* line the exponential becomes purely oscillatory .

$$(1.1) \qquad \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}} + Ce^{-x} \quad (x \to +\infty)$$

What distinguishes the expression (1.1) from a usual asymptotic expansion is the presence beyond all orders of the principal series of an exponentially small term which cannot be captured by classical Poincaré asymptoticity. More general transseries arising in generic ordinary differential and difference equations are doubly infinite series whose terms are powers of x multiplied by exponentials (see e.g. (2.3)). These transseries can be determined algorithmically and usually diverge factorially, but can be shown to be Borel summable in a suitable sense in some sector. In this sector the function to which the transseries sums has good analyticity properties and on the edges of the sector typically singularities appear.

The correspondence between the formal universe and actual functions. Envisaged by Écalle to be a "total" isomorphism implemented by generalized Borel summation, the correspondence has been established rigorously in a number of problems including ODEs, difference equations and PDEs.

2. Difference equations: the isolated movable singularity property (IMSP)

The problem of integrability has been and is a subject of substantial research. In the context of differential equations there exist numerous effective criteria of integrability, among which a crucial role is played by the Painlevé test (for extensive references see e.g. [33]).

An analog of the Painlevé property for difference equations turns out to be more delicate to define.

In the context of solvability, various methods have been proposed by Joshi [38], Ablowitz et al. [1], Ramani and Grammaticos (see references in [36]), Conte and Musette [12]. See also [1] for a comparative discussion of the various approaches in the literature. None of these is a proper extension of the Painlevé test. One difficulty resides in continuing the solutions $x(n)$ of a difference equation, which are defined on a *discrete set*, to the complex plane of the independent variable n in a natural and effective fashion. The embedding of $x(n)$ must be done in such a way that *properties are preserved*. It is important for the effectiveness of the analysis that this embedding $x(n) \sqsubset x(z)$ is natural, constructive and unique under proper conditions.

It is of course crucial that we are given infinitely many values of the function $x(n)$; since the accumulation point of n is infinity, the behavior of x at ∞ is key, and then the question boils down to when infinitely many values determine function, and in which class.

To illustrate a point we start with a rather trivial example. Assume that $x(n)$ is expressed as a convergent power series in powers of $1/n$

$$x_n = \sum_{k=1}^{\infty} c_k n^{-k}$$

We would then naturally define $x(n) \sqsubset x(z)$ by

$$x(z) = \sum_{k=1}^{\infty} c_k z^{-k}$$

for large enough z. Uniqueness is ensured by the analyticity at ∞ of $x(z)$. Still by analyticity, $\sqsubset$ preserves all properties.

We cannot rely merely on analytic continuation since there are very few classes of equations where solutions are given by convergent power series. However, as discovered by Écalle and proved in detail very recently in a general setting by Braaksma [11], a wide class of arbitrary order difference equations admit formal solutions as Borel summable *transseries*. The class considered by Braaksma is of the form

$$(2.1) \qquad \mathbf{x}_{n+1} = \mathbf{G}(\mathbf{x}_n, n) = \left(\hat{\Lambda} + \frac{1}{n} \hat{A} \right) \mathbf{x}_n + \mathbf{g}(n, \mathbf{x}_n)$$

where $\mathbf{G}$ analytic at ∞ in n and at 0 in x_n, under genericity assumptions [11]. In particular a nonresonance condition is imposed

$$(2.2) \qquad \mu_m = \mathbf{k} \cdot \boldsymbol{\mu} \mod 2\pi i$$

with $\mathbf{k} \in \mathbb{N}^n$ iff $\mathbf{k} = \mathbf{e}_m$. Transseries solutions for these equations have many similarities to transseries solutions of differential equations. With some m_1, $0 \leq m_1 \leq n$,

$$(2.3) \qquad \tilde{\mathbf{x}}(n) = \sum_{\mathbf{k} \in \mathbb{N}^{m_1}} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k} \cdot \boldsymbol{\mu} n} n^{\mathbf{k} \cdot \mathbf{a}} \tilde{\mathbf{y}}_{\mathbf{k}}(n)$$

In (2.3) $\boldsymbol{\mu} = (\mu_1, ..., \mu_{m_1})$ and $\mathbf{a} = (a_1, ..., a_{m_1})$ depend only on the recurrence (they are the eigenvalues of $\hat{\Lambda}, \hat{A}$, respectively), $\mathbf{C}$ is a free parameter and $\tilde{\mathbf{y}}_{\mathbf{k}}$ are formal series in negative integer powers of n, independent on $\mathbf{C}$. The number m_1 is chosen so that all the exponentials in (2.3) tend to zero in the chosen sector.

Braaksma showed that $\tilde{\mathbf{y}}_{\mathbf{k}}(n)$ are Borel summable uniformly in $\mathbf{k}$. Let $\mathbf{Y}_{\mathbf{k}} = \mathcal{L}^{-1} \tilde{\mathbf{y}}_{\mathbf{k}}(n)$. Then $\mathbf{Y}_{\mathbf{k}}(p)$ are analytic in a neighborhood of $\mathbb{R}^+$ (in fact in a larger sector). Defining

$$(2.4) \qquad \mathbf{y}_{\mathbf{k}} = \int_0^{\infty} e^{-np} \mathbf{Y}_{\mathbf{k}}(p) dp$$

we have uniform estimates $|\mathbf{y}_{\mathbf{k}}| < A^{\mathbf{k}}$ and thus the series

$$(2.5) \qquad \mathbf{x}(n) = \sum_{\mathbf{k} \in \mathbb{N}^{m_1}} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k} \cdot \boldsymbol{\mu} n} n^{\mathbf{k} \cdot \mathbf{a}} \mathbf{y}_{\mathbf{k}}(n)$$

is classically convergent for large enough n. Braaksma showed that $\mathbf{x}(n)$ is an actual solution of (2.1). It is natural to replace n by z in (2.4) and define:

$$(2.6) \qquad \mathbf{x}(z) = \sum_{\mathbf{k} \in \mathbb{N}^{m_1}} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k} \cdot \boldsymbol{\mu} z} z^{\mathbf{k} \cdot \mathbf{a}} \mathbf{y}_{\mathbf{k}}(z)$$

If z and all constants are real and $\mu_i < 0$, the functions (2.6) are special cases of Écalle's *analyzable* functions. As explained before we are allowing for z and constants to be complex, under restriction $\Re(\mathbf{k} \cdot \boldsymbol{\mu} z) > 0$.

It is crucial that the values of $\mathbf{y}(\mathbf{n})$ for all large enough n uniquely determine the expansion. In [17] it is shown that under suitable conditions, *two distinct analyzable functions cannot agree on a set of points accumulating at infinity.* Below is a simplified version of a theorem in [17].

Theorem 1 ([17]). *Assume*

$$(\mathbf{Z} \cdot \boldsymbol{\mu} = 0 \mod 2\pi i \;\; with \;\; \mathbf{Z} \in \mathbb{Z}^p) \Leftrightarrow \mathbf{Z} = 0$$

and $\mathbf{x}(z) = \sum_{\mathbf{k} \in \mathbb{N}^{m_1}} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k} \cdot \boldsymbol{\mu} z} z^{\mathbf{k} \cdot \mathbf{a}} \mathbf{y_k}(z)$. *If* $\mathbf{x}(n) = 0$ *for all large enough* $n \in \mathbb{N}$, *then* $\mathbf{x}(z)$ *is identically zero.*

Analyzable functions behave in most respects as analytic functions. Among the common properties, particularly important is the uniqueness of the extension from sets with accumulation points, implying the principle of permanence of relations. Under the assumptions [11] that ensure Borel summability, we have the following.

Definition 2. *If*

$$\mathbf{x}(n) = \sum_{\mathbf{k} \in \mathbb{N}^{m_1}} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k} \cdot \boldsymbol{\mu} n} n^{\mathbf{k} \cdot \mathbf{a}} \mathbf{y_k}(n)$$

then we call

$$\mathbf{x}(z) = \sum_{\mathbf{k} \in \mathbb{N}^{m_1}} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k} \cdot \boldsymbol{\mu} z} z^{\mathbf{k} \cdot \mathbf{a}} \mathbf{y_k}(z)$$

the analyzable embedding of $\mathbf{x}(n)$ *from* $\mathbb{N}$ *to a sector in* $\mathbb{C}$.

Having now a suitable procedure of analytic continuation, we can define the isolated movable singularity property, an extension of the Painlevé property to difference equations, in a natural fashion. In analogy with the case of differential equations we require that all solutions are free from "bad" movable singularities.

Definition 3. *A difference equation has the IMSP if all* movable *singularities of all its solutions are* **isolated.**

Notes. (i) We use the common convention that isolated singularity exclude branch points, clusters of poles and barriers of singularities.

(ii) To determine the singularities, transasymptotic matching methods introduced for differential equations in [18], can be extended with little changes to difference equations.

3. CLASSIFICATION OF SOME DIFFERENCE EQUATIONS WITH IMSP. SOLVABILITY.

We look at autonomous difference equations of the form $x_{n+1} = G(x_n)$ where G is meromorphic and has attracting fixed points. A more general analysis is given in [17]. We then write

$$(3.1) \qquad\qquad x_{n+1} = G(x_n) := a x_n + F(x_n)$$

and restrict for simplicity to the case $F(0) = F'(0) = 0$ and $0 < |a| < 1$. There is a one-parameter family of solutions presented as simple transseries *convergent* for large enough n, of the form

$$(3.2) \qquad x_n = x_n(C) = \sum_{k=1}^{\infty} e^{nk \ln a} C^k D_k$$

for given values of D_k, independent of C. The analyzable embedding of x, cf. Definition 2, reads

$$(3.3) \qquad x(z) = x(z; C) = \sum_{k=1}^{\infty} e^{zk \ln a} C^k D_k$$

which is analytic for large enough z. To look for the IMSP, we find the properties of $x(z)$ beyond the domain of convergence of (3.3), and find the singular points of $x(z)$.

Note. Because equation (3.1) is nonlinear, although (3.3) has one continuous parameter, there may be more solutions. This issue is addressed in ([17]).

3.1. **Embedding versus properties of the conjugation map.** By the Poincaré conjugation theorem applied to $x_{n+1} = G(x_n)$ there exists a unique map ϕ with the properties

$$(3.4) \qquad \phi(0) = 0, \quad \phi'(0) = 1 \ \text{ and } \phi \text{ analytic at } 0$$

and such that $x_n = \phi(X_n)$ implies $X_{n+1} = aX_n$. The map ϕ is a conjugation map of x_{n+1} with its linearization X_{n+1}. We have $X_n = a^n X_0 = Ca^n$.

We obtain an extension of x from $\mathbb{N}$ to $\mathbb{C}$ through

$$(3.5) \qquad x_n = \phi(Ca^n) \ \sqsubset \ x(z) := \phi(Ca^z)$$

Then the conjugation map satisfies the equation

$$\phi(az) = G(z) = a\phi(z) + F(\phi(z))$$

As expected by the uniqueness of the embedding in Definition 2 we have the following.

Lemma 4. *(i) For equations of the type (3.1) under the given assumptions, the embeddings through analyzability and via the conjugation map, cf. (3.5) agree.*

(ii) The solutions $x(z; C)$ have only isolated movable singularities iff ϕ has only isolated singularities.

Corollary *A necessary condition for (3.1) to have the IMSP is that the conjugation map ϕ, around every stable fixed point of G and of those of its iterates which are meromorphic (of the form $G^{[m]} = G \circ \cdots \circ G$ m times) extends analytically into the complex plane, except for isolated singularities.*

3.2. **Autonomous equations with the IMSP.** In ([17]) we classify the equations of the form (3.1) with respect to the IMSP. In a way analog to the case of ODEs of first order, only Riccati equations have this property. On the other hand, these equations are explicitly solvable, and thus autonomous equations of the first order which have the IMSP can be solved in closed form.
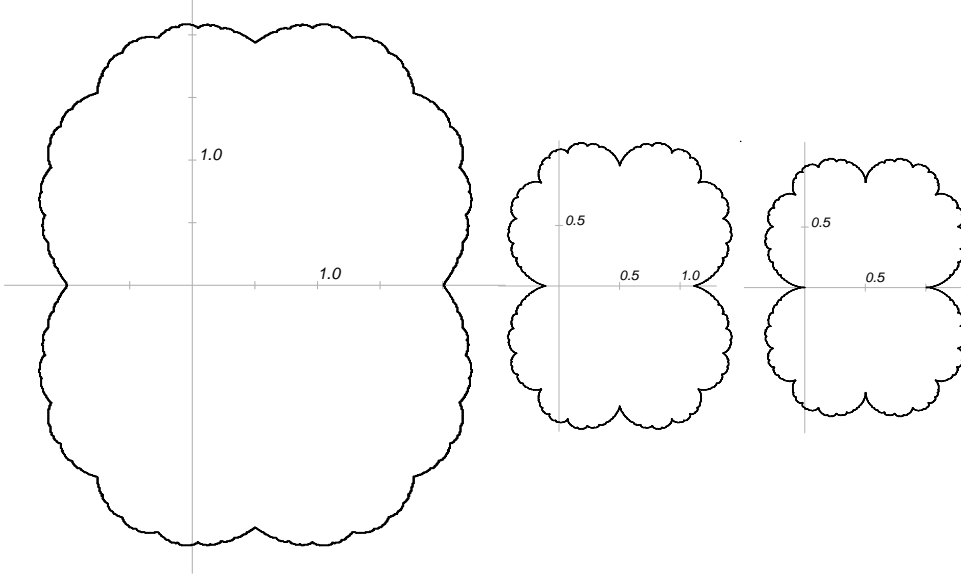
FIGURE 1. Julia sets for $G = ax(1-x)$ for $a = 0.5$, $a = 0.9$ and $a = 1$ respectively. The set $\mathcal{K}_p$ is the interior of these curves (graphs generated using C and MapleV)

Theorem 5. *The equation (3.1) under the given assumptions* fails *to have the IMSP unless, for some $c \in \mathbb{C}$,*

$$(3.6) \qquad G(z) = \frac{az}{1 + cz}$$

In case (3.6 we have: $1/x_n = a^{-n}(C - c/(a-1)) + c/(a-1)$

It is also very interesting to see that in the case of failure of the IMSP the analytic properties of the solutions preclude the existence of nice constants of motion. This is discussed in the next section.

3.3. **Case of failure of IMSP.** We illustrate this situation when G polynomial. The surprising conclusion is that for polynomial G without the IMSP, constants of motion (defined as functions $C(x, n)$ constant along trajectories) develop barriers of singularities along some fractal closed curves $\partial\mathcal{K}_p$, see below. In a neighborhood of the origin, the function ϕ is invertible, and thus for small x we derive from $x_n = \phi(a^n C)$ that $C = a^{-n}Q(x_n)$ where $Q = \phi^{-1}$.

In Fig. 1. we depict the Julia set $\partial\mathcal{K}_p$ of a simple map. In that case, in the compact set bounded by $\mathcal{K}_p$ consists in the initial conditions for which the solution of the iteration converges to zero. The Julia set is a closed curve of nontrivial fractal

dimension. For a comprehensive discussion of Julia sets and iterations of rational maps see [5].

Theorem 6. *Assume G is a polynomial map with an attracting fixed point at the origin. Denote by $\mathcal{K}_p$ (cf. Fig. 1) the maximal connected component of the origin in the Fatou set of G.*

Then the domain of analyticity of Q is $\mathcal{K}_p$, and $\partial\mathcal{K}_p$ is a barrier of singularities of Q.

The proofs of these results can be found in [17]. The logistic map discussed in relative detail in the next section represents a very simple illustration of some of the relevant phenomena.

4. ANALYSIS OF THE LOGISTIC MAP AT THE SUPERSTABLE FIXED POINT INFINITY

We show that the equation $x_{n+1} = ax_n(1-x_n)$ has the IMSP iff $a \in \{-2, 0, 2, 4\}$. The case $a = 0$ needs no analysis. Otherwise, taking $y = 1/x$ we get

$$(4.1) \qquad y_{n+1} = \frac{y_n^2}{a(y_n - 1)}$$

For small y_0, the leading order form of equation (4.1) is $y_{n+1} = -a^{-1}y_n^2$ whose solution is $-y_0^{2^n}a^{-2^{n-1}-1}$. It is then convenient to seek solutions of (4.1) in the form $y_n = F(y_0^{2^n}a^{-2^n})$ whence the initial condition implies $F(0) = 0$, $F'(0) = -a$. Denoting $y_0^{2^n}a^{-2^n} = z$, the functional relation satisfied by F is

$$(4.2) \qquad F(z^2) = \frac{F(z)^2}{a(F(z) - 1)} \qquad (F(0) = 0, \ F'(0) = -a)$$

Lemma 7. *(i) There exists a unique analytic function F in the neighborhood of the origin satisfying (4.2) and such that $F(0) = 0$ and $F'(0) = -a$. This F has only isolated singularities in $\mathbb{C}$ if and only if $a \in \{-2, 2, 4\}$. In the latter case, the equations (4.2) and (4.1) can be solved explicitly (see (4.5), (4.8) and (4.9)).*

If $a \notin \{-2, 2, 4\}$ then the unit disk is a barrier of singularities of F.

(ii) If $a \notin \{-2, 2, 4\}$ then $\partial\mathcal{K}_p$ is a barrier of singularities of the constant of motion $Q = F^{-1}$ (cf. Theorem 6).

4.1. **Proof of Lemma 7.**

4.1.1. *Analyticity at zero.* We write

$$(4.3) \qquad F(z) = \frac{a}{2}F(z^2) - \frac{1}{2}\sqrt{a^2F(z^2)^2 - 4aF(z^2)}$$

(with the choice of branch consistent with $F(0) = 0, F'(0) = -a$), and take $F(z) = -az + h(z)$. This leads to the equation for h

$$(4.4) \quad h(z) = az + \frac{a}{2}(h(z^2) - az^2) - \frac{1}{2}\sqrt{4a^2z^2 - 4ah(z^2) + a^2(h(z^2) - az^2)^2} = \mathcal{N}(h)$$

It is straightforward to show that, for small ϵ, $\mathcal{N}$ is contractive in the ball of radius $|a|^2$ in the space of functions of the form $h(z) = z^2u(z)$, where u is analytic in the disk $D = \{z : \|z\| < \epsilon\}$ with the norm $\|h\| = \sup_D |u|$. The corresponding

F is analytic for small z and is a conjugation map between (4.1) and its small-y approximation.

4.1.2. Now the question is to determine the singularities of F in the complex plane. Inside the unit disk we can inductively continue F analytically through (4.3) as follows. If F is analytic in a disk of radius $r < 1$ then (4.3) provides the analytic continuation in the disk of radius $\sqrt{r}$ if $F(z^2)$ is not zero or $4/a$. Because F is assumed to be analytic in the disk of radius r, $F(z^2)$ cannot vanish; otherwise by (4.2) it would vanish infinitely often in a neighborhood of zero, which is inconsistent with its analyticity at $z = 0$ and with $F'(0) = -a$. But the equation $F(z^2) = 4/a$ does have solutions in the unit disk if $|a| > 5$. Indeed, with $|a| > 5$ it is immediate to show $F(z^2) = 4/a$ implies $F(z^{2^n}) := \epsilon_n \to 0$ as $n \to \infty$. Since F is, by the condition $F'(0) = -a$, bijective near $z = 0$, the equation $F(z) = \epsilon_n$ has a (unique) solution, $z = z_0$ for sufficiently large n. Now, if z_1 is such that $z_1^{2^n} = z_0$ we have $F(z_0) = 4/a$. But then F has a square root branch point at z_1 since $F'(z)$ cannot vanish anywhere inside the unit disk, otherwise again by (4.2), $F'(z)$ would vanish infinitely often near the origin.

Thus for $|a| > 5$ there is no F with only isolated singularities. In fact, as is seen in the last paragraph of the proof, in this case the unit disk is a barrier of singularities of F.

We now consider the case where $|a| < 5$.

4.1.3. We claim that unless $a = -2$, the point $z = 1$ is a singular point of F. Indeed, a Taylor series expansion $F = \sum_{k=0}^{\infty} c_k(z-1)^k$ gives $c_0 = 0$ or $c_0 = a/(a-1)$. It is straightforward to see that $c_0 = 0$ implies $c_k = 0$ for all k which contradicts $F'(0) = -a$.

Therefore $c_0 = a/(a-1)$ in which case direct calculation shows that, unless $a = -2(2^k - 1)$ for some $k \in \mathbb{Z}$, all c_k for $k \geq 1$ are zero which is not possible since $F(0) = 0$. If $k > 1$ then $|a| > 5$.

Therefore, $k = 1$ whence $a = -2$. With this value of a, (4.2) has the explicit solution

$$(4.5) \qquad F(z) = \frac{2z}{z^2 + z + 1}$$

It remains to look at the cases when $z = 1$ is a singular point of F.

4.1.4. If $z = 1$ is a meromorphic point of F, then we obtain from (4.2):

$$(4.6) \qquad \lim_{z \to 1} \frac{aF(z^2)}{F(z)} = 1$$

If $F(z) = \sum_{k=-p}^{\infty} c_k(z-1)^k$ is the Laurent series of F at $z = 1$, then (4.6) implies

$$(4.7) \qquad a = 2^p$$

and, since $|a| < 5$, we must have $a \in \{2, 4\}$.
For $a = 2$ (4.2) has the explicit solution

$$(4.8) \qquad F(z) = \frac{2z}{z - 1}$$

For $a = 4$ the solution of (4.2) is

$$(4.9) \qquad F(z) = -\frac{4z}{(z-1)^2}$$

4.1.5. If F is not meromorphic at $z = 1$ then F cannot be continued analytically beyond the unit circle:

Assume first that F is analytic in the open unit disk D_1. By (4.2) we have

$$(4.10) \qquad F(z^{2^n}) = R_n(F(z))$$

where R_n is a rational function. Thus if $z_0^{2^n} = 1$ and F is analytic at z_0 then it is meromorphic at $z = 1$, while the set of points such that $z_0^{2^n} = 1$ is dense on the unit circle.

The last case to analyze is $R < 1$, where R is the (maximal) radius of analyticity of F. Equation (4.3) shows that on the disk of radius R the singularities of F are square root branch points. On the other hand, if $z_0 \in D_1$ is an algebraic branch point and $z_1^2 = z_0$ then z_1 is also an algebraic branch point as can be seen by continuing (4.2) around a small circle near the branch point $z^2 = z_0$. It is then easy to see that if $R < 1$ the branch points accumulate densely towards the unit circle. $\square$

(ii) The proof follows the lines of **??** so we only sketch the main steps. We note that by (i) F is invertible near the origin, so Q is analytic near the origin, where it satisfies the relation $Q^2(z) = Q(z^2/(az-1)$.

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